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ABOUT TWO MANNERS OF USE WEIGHTED LEAST SQUARES APPROACH IN FUZZY STATISTICS

[1]Ciprian Costin POPESCU, [2]Supian SUDRADJAT

[1]Department of Mathematics, Academy of Economic Studies Bucharest
Calea Dorobanţilor, nr. 15-17, Sector 1, Bucharest, Romania
e-mail: cippx@yahoo.com

[2]Faculty of Mathematics and Natural Sciences, Padjadjaran University, Bandung, Indonesia
e-mail: adjat03@yahoo.com

Abstract. Two fuzzy statistical models are generalized and extended to the weighted models. The new regression problems are solved and compared with the originals.

Key words : fuzzy number; regression; weighted distance; weighted least squares.

1. Introduction

This work tackles the parameter estimation for regression problems with fuzzy data. Generally, the estimation problems consist in choosing and minimizing an objective function (see Arthanary and Dodge [1]). One of the most used methods in this domain is fuzzy least squares (Diamond [5,6,7] and Coppi [4] for triangular fuzzy numbers; Ming, Friedman and Kandel [11] in a more general case). This paper brings together two models, consequently two approaches for this subject: first, proposed by Coppi et al. [4] and the second, given by Ming et al. [11]. We generalize the two norms in each case and obtain the weighted models. This kind of weighted models describes with more accuracy various phenomenons (for example a lot of economic models). The estimations for parameters and the regression lines in each distinct situation are given.

2. The Weighted Coppi Model

We generalize and test the viability of the algorithm developed by R . Coppi et al. [4] for estimation problems which implies fuzzy data.

Let the input fuzzy variables $X_1,\ldots,X_m$ and a fuzzy output variable, $\overline{Y}$, on a size n sample [4]. The data will be denoted by $(\tilde{y}_i, x_i),\ i=\overline{1,n}$, where $x_i^T=(x_{i_1},\ldots,x_{m_1})$. We work with LR fuzzy variables: $\tilde{Y}=(m,l,u)_{LR}$, where

$$\mu(y)=\begin{cases} L\left(\dfrac{m-y}{l}\right), & y\le m:(l>0),\\ R\left(\dfrac{y-m}{u}\right), & y\ge m:(u>0).\end{cases}$$

We consider the theoretical values $\mu, \underline{\partial_L}, \underline{\partial_U}$ and the errors $\underline{\varepsilon}, \underline{\varepsilon_L}, \underline{\varepsilon_U}$. Thus we may write

$$\begin{aligned} m &= \mu + \underline{\varepsilon} \\ m - l &= (\mu - \underline{\partial_L}) + \underline{\varepsilon_L} \\ m + u &= (u - \underline{\partial_U}) + \underline{\varepsilon_U} \end{aligned} \tag{2.1}$$

Or, after a reparametrization:

$$\begin{aligned} \mu &= F_{\underline{\gamma}} \\ \underline{\partial_L} &= \eta_L \mu + \underline{\varepsilon_L} \mathbf{1} \\ \underline{\partial_U} &= \eta_U \mu + \underline{\varepsilon_U} \mathbf{1} \end{aligned} \tag{2.2}$$

where F is a matrix, whose rows have the form $f_i^T = \left[f_1(x_i), \ldots, f_p(x_i)\right]$. Thus the theoretical values of the output variable are $\tilde{y}_i^* = \left(\mu_i, \underline{\partial_{L_i}}, \underline{\partial_{U_i}}\right)_{LR}$, $i = \overline{1, n}$.

Definition 2.1

We introduce the following weighted distance which is a generalization of a metric proposed by Coppi and d'Urso [4]*: for* $w = (w_1, w_2, w_3)$ *we have*

$$\begin{aligned} d^2(w; \tilde{y}; \tilde{y}^*) = d^2(w; (m,l,u)_{LR}, (\mu, \underline{\partial_L}, \underline{\partial_U})_{LR}) = \Delta^2_{LR}(w; \cdot) \\ = w_1 \|m - \mu\|^2 + w_2 \|(m - \lambda l) - (\mu - \lambda \underline{\partial_L})\|^2 + \\ w_3 \|(m - \rho u) - (\mu - \rho \underline{\partial_U})\|^2 \end{aligned}.$$

Theorem 2.1

The following relation holds:

$$\begin{aligned} d^2(w; \tilde{y}; \tilde{y}^*) = (w_1, w_2, w_3)[(m - \mu)^T (m - \mu)] - 2 w_2 \lambda (m - \mu)^T (l - \underline{\partial_L}) + \\ w_2 \lambda^2 (l - \underline{\partial_L})^T (l - \underline{\partial_L}) + 2 w_3 \rho (m - \mu)^T (\mu - \underline{\partial_U}) + \\ w_3 \rho^2 (u - \underline{\partial_U})^T (u - \underline{\partial_U}). \end{aligned}$$

Proof:

$$\begin{aligned} d^2(w; \tilde{y}; \tilde{y}^*) = \Delta^2_{LR}(w; \cdot) = w_1 \|m - \mu\|^2 + w_2 \|(m - \mu) - \lambda (l - \underline{\partial_L})\|^2 + \\ w_3 \|(m - \mu) + \rho (u - \underline{\partial_U})\|^2 \\ = w_1 (m - \mu)^T (m - \mu) + \\ [w_2 (m - \mu)^T (m - \mu) - 2 w_2 \lambda (m - \mu)^T (l - \underline{\partial_L}) + \\ w_2 \lambda^2 (l - \underline{\partial_L})^T (l - \underline{\partial_L})] + \\ [w_3 (m - \mu)^T (m - \mu) + 2 w_3 \rho (m - \mu)^T (u - \underline{\partial_U}) + \\ w_3 \rho^2 (u - \underline{\partial_U})^T (u - \underline{\partial_U}) \end{aligned}$$

$$= (w_1 + w_2 + w_3)\big[(m-\mu)^T(m-\mu)\big] -$$
$$- 2w_2\lambda(m-\mu)^T(l-\underline{\partial_L}) + w_2\lambda^2(l-\underline{\partial_L})^T(l-\underline{\partial_L})] +$$
$$2w_3\rho(m-\mu)^T(u-\underline{\partial_U}) + w_3\rho^2(u-\underline{\partial_U})^T(u-\underline{\partial_U}).$$

For particular case $w_1 = w_2 = w_3 = 1$ we obtain the results from Coppi and d'Urso [4].

Remark 2.1

The algorithm for finding $\underline{\gamma}, \eta_L, \eta_U, \xi_L, \xi_U$ *is reducing to minimizing the weighted distance* d_w^2 *between the experimental measurements of the response variable* $\tilde{y}_i$, $i = \overline{1,n}$ *and the theoretical values* $\tilde{y}_i^*$. *In other words, we have to solve the problem:*

$$\min_{\underline{\gamma},\eta_L,\eta_U,\varepsilon_L,\varepsilon_U} \Delta^2_{LR}(w;\underline{\gamma},\eta_L,\eta_U,\xi_L,\xi_U).$$

Theorem 2.2

The problem $\min_{\underline{\gamma},\eta_L,\eta_U,\varepsilon_L,\varepsilon_U} \Delta^2_{LR}(w;\underline{\gamma},\eta_L,\eta_U,\varepsilon_L,\varepsilon_U)$ *admits local solutions (local minimum) which may be improved using an iterative estimation algorithm.*

Proof:

We have:

$$\Delta^2_{LR}(w;\underline{\gamma},\eta_L,\eta_U,\varepsilon_L,\varepsilon_U) = (w_1,w_2,w_3)[(m-F\underline{\gamma})^T(m-F\underline{\gamma}) -$$
$$2w_2\lambda(m-F\underline{\gamma})^T(1-F\underline{\gamma}\eta_L - \mathbf{1}\xi_L) +$$
$$w_2\lambda^2(1-F\underline{\gamma}\eta_L - \mathbf{1}\xi_L)^T(1-F\underline{\gamma}\eta_L - \mathbf{1}\xi_L) +$$
$$2w_3\rho(m-F\underline{\gamma})^T(u-F\underline{\gamma}\eta_U - \mathbf{1}\xi_U) +$$
$$w_3\rho^2(u-F\underline{\gamma}\eta_U - \mathbf{1}\xi_U)^T(u-F\underline{\gamma}\eta_U - \mathbf{1}\xi_U)$$
$$= (w_1,w_2,w_3)[(m^T - 2m^TF\underline{\gamma} + \gamma^TF^TF\underline{\gamma}) -$$
$$2w_2\lambda(m^Tl - m^TF\underline{\gamma}\eta_L - m^T\mathbf{1}\xi_L - \underline{\gamma}^TF^{T_l} + \underline{\gamma}^TF\underline{\gamma}\eta_L + \underline{\gamma}^TF^T\mathbf{1}\varepsilon_L) +$$
$$w_2\lambda^2(l^T - 2l^TF_{\underline{\gamma}\eta_L} - 2l^T\mathbf{1}\varepsilon_L + \underline{\gamma}^TF^TF\underline{\gamma}\eta_L^2 + 2\underline{\gamma}^TF^T\mathbf{1}\eta_L\xi_L + \eta\xi_L^2) +$$
$$2w_3\rho(m^Tu - m^TF\underline{\gamma}\eta_U - m^T1\varepsilon_U - \underline{\gamma}^TF^Tu + \underline{\gamma}^TF^TF\underline{\gamma}\eta_U + \underline{\gamma}^TF^T\mathbf{1}\varepsilon_u^2) +$$
$$w_3\rho^2(u^Tu - 2u^TF\underline{\gamma}\eta_U - 2u^T\mathbf{1}\varepsilon_U + \underline{\gamma}^TF^TF\underline{\gamma}\eta_U^2 + 2\underline{\gamma}^TF^T\mathbf{1}\eta_U\varepsilon_U + n\xi_U^2).$$

We equate to zero the partial derivatives of $\Delta^2_{LR}(w;\eta_L,\eta_U,\varepsilon_L,\varepsilon_U,\underline{\gamma})$:

$$w_2[\underline{\gamma}^TF^Tm - \underline{\gamma}^TF^TF\underline{\gamma} - \lambda(\underline{\gamma}^TF^Tl - \underline{\gamma}^TF^TF\underline{\gamma}\eta_L - \underline{\gamma}^TF\mathbf{1}\xi_L)] = 0 \qquad (2.1)$$

$$w_3[-\underline{\gamma}^TF^Tm + \underline{\gamma}^TF^TF\underline{\gamma} - \rho(\underline{\gamma}^TF^Tu - \underline{\gamma}^TF^TF\underline{\gamma}\eta_U - \underline{\gamma}^TF^T\mathbf{1}\xi_U)] = 0 \qquad (2.2)$$

$$w_2[\underline{\gamma}^T F^T \mathbf{1} - m^T \mathbf{1} + \lambda(l^T \mathbf{1} - \underline{\gamma}^T F^T F \mathbf{1}\eta_L - n\xi_L)] = 0 \tag{2.3}$$

$$w_3[\underline{\gamma}^T F^T \mathbf{1} + m^T \mathbf{1} - \rho(u^T \mathbf{1} - \underline{\gamma}^T F^T F \mathbf{1}\eta_U - n\xi_U)] = 0 \tag{2.4}$$

$$\begin{aligned} &F^T F \underline{\gamma}^T[(w_1 + w_2 + w_3) - 2w_2\lambda\eta_L + w_2\lambda^2\eta_L^2 + 2w_3\rho\eta_U + w_3\rho^2\eta_U^2] = \\ &(w_1 + w_2 + w_3)F^T m - w_2\lambda(F^T m\eta_L + F^T l - F^T \mathbf{1}\xi_L) + \\ &w_3\rho^2(F^T u\eta_U - F^T \mathbf{1}\eta_U\xi_U) \end{aligned} \tag{2.5}$$

An iterative solution is given by relations (2.6)-(2.10):

$$\eta_L = \lambda^{-1}(\underline{\gamma}^T F^T F\underline{\gamma})^{-1}[\lambda(\underline{\gamma}^T F^T l - \underline{\gamma}^T F^T \mathbf{1}\xi_L) - (\underline{\gamma}^T F^T m - \underline{\gamma}^T F^T F\underline{\gamma})] \tag{2.6}$$

$$\eta_U = \rho^{-1}(\underline{\gamma}^T F^T F\underline{\gamma})^{-1}[\rho(\underline{\gamma}^T F^T u - \underline{\gamma}^T F^T \mathbf{1}\xi_U) + (\underline{\gamma}^T F^T m - \underline{\gamma}^T F^T F\underline{\gamma})] \tag{2.7}$$

$$\xi_L = (n\lambda)^{-1}[\lambda\mathbf{1}^T(l - F\underline{\gamma}\eta_L) - \mathbf{1}^T(m - F\underline{\gamma})] \tag{2.8}$$

$$\xi_U = (n\rho)^{-1}[\rho\mathbf{1}^T(u - F\underline{\gamma}\eta_U) + \mathbf{1}^T(m - F\underline{\gamma})] \tag{2.9}$$

$$\begin{aligned} &\underline{\gamma} = [(w_1 + w_2 + w_2) - w_2\lambda\eta_L(2 - \lambda\eta_L) + w_3\rho\eta_U(2 + \rho\eta_U)]^{-1} \cdot \\ &(F^T F)^{-1} F^T) \times \\ &[(w_1 + w_2 + w_3)m - w_2\lambda(m\eta_L + l - \mathbf{1}\xi_L) + w_2\lambda^2(l\eta_L - \mathbf{1}\eta_L\xi_L) + \\ &w_3\rho(m\eta_U + u - \mathbf{1}\xi_U) + w_3\rho^2(u\eta_U - \mathbf{1}\eta_U\xi_U)]. \end{aligned} \tag{2.10}$$

We don't have the certainty that the equations (2.6)-(2.10) give a global solution. It's necessary to resort to an iterative algorithm. The routine for finding the iterative solution with help of a computer is available in literature (see Coppi et al.).

Remark 2.2

After some calculation, we observe that the properties of the solution obtained from the weighted model are the same as in the Coppi's nonweighted model (see Coppi et al. [4], Proposition 1,2,3):

i) $w_1\mathbf{1}^T(m - \hat{\mu}) = 0;\ w_2\mathbf{1}^T(l - \underline{\hat{\partial}_L}) = 0;\ w_3\mathbf{1}^T(u - \underline{\hat{\partial}_U}) = 0;$

ii) $w_1(m - \hat{\mu})^T = 0;$

iii) $(l - \underline{\hat{\partial}_L})^T \underline{\hat{\partial}_L} = 0;\ (u - \underline{\hat{\partial}_U})^T \underline{\hat{\partial}_U} = 0;$

where $\hat{\mu}, \underline{\hat{\partial}_L}, \underline{\hat{\partial}_U}$ are the iterative solutions obtained from the normal equations system (2.6)-(2.10).

3. An Extension for The Ming Approach

In this section, a weighted model especially based on Ming approach in the field of fuzzy optimization, is developed.

Definition 3.1.

We consider a fuzzy space F^1 *as a function space with the following properties* [2,11]*:*

i) *The elements of* F^1 *are the functions* $f : R \to [0,1]$ *, which are called fuzzy numbers;*

ii) $f(x_0) = 1$ *for some* $x_0 \in R$*;*

iii) $f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\}, x, y \in R, \alpha \in [0,1]$;

iv) $\lim_{x \to t} \sup f(x) = f(t), t \in R$*;*

v) $[f]^0 = closere\{t / t \in R, f(t) > 0\}$ *is compact.*

Definition 3.2

If $f, g \in F^1, \beta \in R, r \in [0,1]$ *we may introduce* [10,12]:

$$[f]^r = \begin{cases} \left\{ \dfrac{t}{f(t)} \geq r \right\}, 0 < r \leq 1; \\ \left\{ \dfrac{t}{f(t)} > 0 \right\}, r = 0; \end{cases}$$

consequenctly, the operations with fuzzy numbers are:

i) $[f]^r + [g]^r = \{a + b / a \in [f]^r, b \in [g]^r\};$*;*
ii) $\beta[f]^r = \{\beta a / a \in [f]^r\}$.

Remark 3.1

$[f]^r = [\underline{f}(r), \overline{f}(r)]$ *is a closed interval and* [8,11]*:*

i) $\underline{f}(r)$ *is a bounded left continuous nondecreasing function over* [0,1]*;*

ii) $\overline{f}(r)$ *is a bounded left continuous nonincreasing function over [0,1];*

iii) $\underline{f}(r) \leq \overline{f}(r)$ *, for all* $r \in [0,1]$*;*

iv)the functions $[\underline{f}(r), \overline{f}(r)]$ *define a unique fuzzy number* $f \in F^1$.

Remark 3.2

Since $X_i = \left(\underline{X_i}(r), \overline{X_i}(r)\right) \in F^1$ *then the triangular form for* X_i *is:*

$X_i = (x_i, \underline{u_i}, \overline{u_i})$ *where* $\underline{X_i}(r) = x_i - \underline{u_i} + \underline{u_i} r$ *and* $\overline{X_i}(r) = x_i + \overline{u_i} - \overline{u_i} r$ [11].

Definition 3.3

Now we consider a generalization of metric $D_2[2,3,8,9,11]$*, in fact a weighted metric:*
for $f, g, w \in F^1$ *we define the weighted distance between f and g as follows:*

$$D^2(w;f,g)=\int_0^1 \underline{w}(r)\left(\underline{f}(r)-\underline{g}(r)\right)^2dr+\int_0^1 \overline{w}(r)\left(\overline{f}(r)-\overline{g}(r)\right)^2dr\,.$$

For $X_i=\left(\underline{X}_i(r),\overline{X}_i(r)\right)\in F^1$ *input independent variables,* $\underline{Y}_i=\left(\underline{Y}_i(r),\overline{Y}_i(r)\right)\in F^1$ *response variables and* $w_i=\left(\underline{w}_i(r),\overline{w}_i(r)\right)\in F^1,\ i=\overline{1,n}$, *we have:*

$X_i=\left(x_i,\underline{u}_i,\overline{u}_i\right),\ Y_i=\left(y_i,\underline{v}_i,\overline{v}_i\right),\ w_i=\left(z_i,\underline{t}_i,\overline{t}_i\right)$ *where*

$\underline{X}_i(r)=x_i-\underline{u}_i+\underline{u}_ir,\ \overline{X}_i(r)=x_i-\overline{u}_i+\overline{u}_ir,$

$\underline{Y}_i(r)=y_i-\underline{v}_i+\underline{v}_ir,\ \overline{Y}_i(r)=y_i-\overline{v}_i+\overline{v}_ir,$

$\underline{w}_i(r)=z_i-\underline{t}_i+\underline{t}_ir,\ \overline{w}_i(r)=z_i-\overline{t}_i+\overline{t}_ir.$

We assume that the dependence between X and Y is given by $Y=a+bX$; *a*, *b* are unknown real parameters and $b\neq 0$. It is necessary to estimate *a*, *b* using weighted least squares method.

Thus we must minimize the sum of squared deviations between theoretical and experimental values:

$$S(a,b)=\sum_{i=1}^{n}D^2(w_i;a+bX_i,Y_i)\,.$$

Theorem 3.1

The sum of squared deviations S(a,b) depends on sign of real parameter b.

Proof:

We approximate $\int_0^1 f(r)dr\simeq\frac{f(0)+f(1)}{2}$ (see [6,11]).

If *b>0* we have

$$S_1(a,b)=\sum_{i=1}^{n}\left[\int_0^1 \underline{w_i}(r)(a+b\underline{X}_i(r)-\underline{Y_i}(r))^2dr+\int_0^1 \overline{w_i}(r)(a+b\overline{X}_i(r)-\overline{Y_i}(r))^2dr\right]$$

$$=\sum_{i=1}^{n}\left[\int_0^1 (z_i-\underline{t_i}+\underline{t_i}r)[a+b(x_i-\underline{u_i}+\underline{u_i}r)-y_i+\underline{v_i}-\underline{v_i}r]^2dr\right]+$$

$$\sum_{i=1}^{n}\left[\int_0^1 (z_i+\overline{t_i}-\overline{t_i}r)[a+b(x_i+\overline{u_i}-\overline{u_i}r)-y_i-\overline{v_i}+\overline{v_i}r]^2dr\right]\simeq$$

$$\frac{1}{2}\sum_{i=1}^{n}\Big\{(z_i-\underline{t_i})[a+b(x_i-\underline{u_i})-(y_i-\underline{v_i})]^2+$$

$$(z_i+\overline{t_i})[a+b(x_i+\overline{u_i})-(y_i+\overline{v_i})]^2+2z_i(a+bx_i-y_i)^2\Big\}.$$

If $b<0$ we obtain

$$S_2(a,b)=\sum_{i=1}^{n}\left[\int_0^1 \underline{w_i}(r)(a+b\overline{X}_i(r)-\underline{Y_i}(r))^2dr+\int_0^1 \overline{w_i}(r)(a+b\underline{X}_i(r)-\overline{Y_i}(r))^2dr\right]=$$

$$= \sum_{i=1}^{n}\left[\int_0^1 (z_i - \underline{t_i} + \underline{t_i}r)[a + b(x_i - \underline{u_i} - \underline{u_i}r) - y_i - \underline{v_i} - \underline{v_i}r]^2 d2 + dr\right] +$$

$$\sum_{i=1}^{n}\left[\int_0^1 (z_i + \bar{t}_i - \bar{t}_i r)[a + b(x_i - \underline{u}_i + \underline{u}_i r) - y_i - \bar{v}_i + \bar{v}_i r]^2 dr\right] \underline{\simeq}$$

$$\frac{1}{2}\sum_{i=1}^{n}\Big\{(z_i - \underline{t_i})[a + b(x_i + \overline{u}_i) - (y_i - \underline{v_i})]^2 +$$

$$(z_i + \bar{t}_i)[a + b(x_i - \underline{u_i}) - (y_i + \bar{v}_i)]^2 + 2z_i(a + bx_i - y_i)^2\Big\}.$$

In conclusion, if $b > 0$ then $S(a,b) = S_1(a,b)$; if $b < 0$ then $S(a,b) = S_2(a,b)$.

Theorem 3.2

The problem $\min_{a\in R, b\in R^*} S(a,b)$ *has a unique solution* $(\bar{a},\bar{b})$. *Consequently, there exists a single line, the best line* $Y = \bar{a} + \bar{b}X$ *which fit the given data* (X_i, Y_i).

Proof:

We have the problem

$$\min_{a\in R, b\in R^*} S(a,b).$$

Accordingly to Theorem 3.1 we must discuss two possibilites:

1) $b > 1$. After equate with zero the partial derivatives of $S(a,b)$ we obtain the system:

$$a\sum_{i=1}^{n}[4z_i + \bar{t}_i - \underline{t_i}] + b\sum_{i=1}^{n}[(z_i - \underline{t_i})(x_i - \underline{u_i}) + (z_i - \bar{t}_i)(x_i - \bar{u}_i) + 2z_i x_i] =$$

$$\sum_{i=1}^{n}[(z_i - \underline{t_i})(y_i - \underline{v_i}) + (z_i + \bar{t}_i)(y_i + \bar{v}_i) + az_i y_i]$$

$$a\sum_{i=1}^{n}[(z_i - \underline{t_i})(x_i - \underline{u_i}) + (z_i - \bar{t}_i)(x_i - \bar{u}_i) + 2z_i x_i] +$$

$$b\sum_{i=1}^{n}[(z_i - \underline{t_i})(x_i - \underline{u_i})^2 + (z_i - \bar{t}_i)(x_i - \bar{u}_i)^2 + 2z_i x_i^2] =$$

$$\sum_{i=1}^{n}[(z_i - \underline{t_i})(x_i - \underline{u_i})(y_i - \underline{v_i}) + (z_i + \bar{t}_i)(x_i + \bar{u}_i)(y_i + \bar{v}_i) + 2z_i x_i y_i].$$

The determinant is:

$$\Delta_1 = \left\{\sum_{i=1}^{n}[(z_i - \underline{t}_i) + (z_i - \overline{t}_i) + 2z_i]\right\}\cdot$$
$$\left\{\sum_{i=1}^{n}[(z_i - \underline{t}_i)(x_i - \underline{u}_i)^2 + (z_i + \overline{t}_i)(x_i + \overline{u}_i)^2 + 2z_i x_i^2]\right\} -$$
$$\left\{\sum_{i=1}^{n}[(z_i - \underline{t}_i)(x_i - \underline{u}_i) + (z_i - \overline{t}_i)(x_i + \overline{u}_i) + 2z_i x_i]\right\}^2 \geq 0,$$

from Cauchy-Schwarz inequality. We may $\Delta_1 > 0$ (because X_i are independent variables), and obtain a unique minimization point (a_1, b_1) for $S_1(a,b)$:

$$a_1 = \frac{\Delta_{a_1}}{\Delta_1},\ b_1 = \frac{\Delta_{b_1}}{\Delta_1} \text{ where}$$
$$\Delta_{a_1} = \left\{\sum_{i=1}^{n}[(z_i - \underline{t}_i)(y_i - \underline{v}_i) + (z_i + \overline{t}_i)(y_i + \overline{v}_i)^2 + 2z_i y_i]\right\}\cdot$$
$$\left\{\sum_{i=1}^{n}[(z_i - \underline{t}_i)(x_i - \underline{u}_i)^2 + (z_i - \overline{t}_i)(x_i + \overline{u}_i) + 2z_i x_i^2]\right\} -$$
$$\left\{\sum_{i=1}^{n}[(z_i - \underline{t}_i)(x_i - \underline{u}_i)(y_i - \underline{v}_i) + (z_i + \overline{t}_i)(x_i + \overline{u}_i)(y_i + \overline{v}_i) + 2z_i x_i y_i]\right\}\cdot$$
$$\left\{\sum_{i=1}^{n}[(z_i - \underline{t}_i)(x_i - \underline{u}_i) + (z_i + \overline{t}_i)(x_i + \overline{u}_i) + 2z_i x_i]\right\} \text{ and}$$
$$\Delta_{b_1} = \left\{\sum_{i=1}^{n}[4z_i + \overline{t}_i - \underline{t}_i)\ ]_i\right\}\cdot$$
$$\left\{\sum_{i=1}^{n}[(z_i - \underline{t}_i)(x_i - \underline{u}_i)(y_i - \underline{v}_i) + (z_i + \overline{t}_i)(x_i + \overline{u}_i)(y_i + \overline{v}_i) + 2z_i x_i y_i]\right\} -$$
$$\left\{\sum_{i=1}^{n}[(z_i - \underline{t}_i)(x_i - \underline{u}_i) + (z_i + \overline{t}_i)(x_i + \overline{u}_i) + 2z_i x_i y_i]\right\}\cdot$$
$$\left\{\sum_{i=1}^{n}[(z_i - \underline{t}_i)(y_i - \underline{v}_i) + (z_i + \overline{t}_i)(y_i + \overline{v}_i) + 2z_i y_i]\right\}.$$

2) $b < 0$. We have:

$$a\sum_{i=1}^{n}[4z_i + \overline{t}_i - \underline{t}_i] + b\sum_{i=1}^{n}[(z_i - \underline{t}_i)(x_i + \overline{u}_i) + (z_i + \overline{t}_i)(x_i - \underline{u}_i) + 2z_i x_i] =$$

$$\sum_{i=1}^{n}[(z_i-\underline{t}_i)(y_i-\underline{v}_i)+(z_i+\overline{t}_i)(y_i+\overline{v}_i)+2z_iy_i]$$

$$a\sum_{i=1}^{n}[(z_i-\underline{t}_i)(x_i-\overline{u}_i)+(z_i-\overline{t}_i)(x_i-\underline{u}_i)+2z_ix_i]+$$

$$b\sum_{i=1}^{n}[(z_i-\underline{t}_i)(x_i-\overline{u}_i)^2+(z_i-\overline{t}_i)(x_i-\underline{u}_i)^2+2z_ix_i^2]=$$

$$\sum_{i=1}^{n}[(z_i-\underline{t}_i)(x_i-\overline{u}_i)(y_i-\underline{v}_i)+(z_i+\overline{t}_i)(x_i+\underline{u}_i)(y_i+\overline{v}_i)+2z_ix_iy_i].$$

Now,

$$\Delta_2=\left\{\sum_{i=1}^{n}[4z_i+\overline{t}_i-\underline{t}_i]\right\}\left\{\sum_{i=1}^{n}[(z_i-\underline{t}_i)(x_i+\overline{u}_i)^2+(z_i+\overline{t}_i)(x_i+\underline{u}_i)^2+2z_ix_i^2]\right\}-$$

$$\left\{\sum_{i=1}^{n}[(z_i-\underline{t}_i)(x_i-\underline{u}_i)+(z_i+\overline{t}_i)(x_i+\underline{u}_i)+2z_ix_i]\right\}^2\geq 0.$$

As in above case we obtain that this system has a unique minimization point (a,b).

If $S(a_1,b_1)<S(a_2,b_2)$ then $(\overline{a},\overline{b})<(a_1,b_1)$.
If $S(a_1,b_1)>S(a_2,b_2)$ then $(\overline{a},\overline{b})=(a_2,b_2)$.

Thus we obtain a unique solution for estimation problem. The theorem was proved.

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